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FUNCTION THEORY ON THE NEILE PARABOLA

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Abstract. We give a formula for the Carathéodory distance on the Neile Parabola $\{(z, w) \in \mathbb{D}^2 : z^2 = w^3\}$ restricted to the bidisk, making it the first variety with a singularity to have its Carathéodory pseudo-distance explicitly computed. This addresses a recent question of Jarnicki and Pflug. In addition, we relate this problem to a mixed Carathéodory-Pick interpolation problem for which known interpolation theorems do not apply. Finally, we prove a bounded holomorphic function extension result from the Neile parabola to the bidisk.

1. Introduction

Distances on a complex space X which are invariant under biholomorphic maps have played an important role in the geometric approach to complex analysis. One of the oldest such distances is the the Carathéodory pseudo-distance c_X ("pseudo" because the distance between two points can be zero). It was introduced by C. Carathéodory in 1926 and is extremely simple to define. The distance between two points x and y is defined to be the largest distance (using the Poincaré hyperbolic distance) that can occur between $f(x)$ and $f(y)$ under a holomorphic map f from X to the unit disk $\mathbb{D} \subset \mathbb{C}$. The Kobayashi pseudo-distance k_X, introduced by S. Kobayashi in 1967, is defined in the opposite direction: the "distance" between two points x and y is now the infimum of the (hyperbolic) distance that can occur between two points $a, b \in \mathbb{D}$ for which there is a holomorphic map f from the disk to X mapping a to x and b to y. (Actually, there is a small technicality here—see section 4 for the true definition). A consequence of the Schwarz-Pick lemma on the disk (which says holomorphic self-maps of the disk are distance decreasing in the hyperbolic distance) is the fact that $c_X \leq k_X$.

For the purposes of motivating the present article, let us indulge in a short tangent. An interesting question, because of its geometric implications, is for which complex spaces do we have $c_X = k_X$? The most important contribution to this question is by L. Lempert [10].

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Lempert's theorem proves the Carathéodory and Kobayashi distances agree on a convex domain. This theorem came as a surprise for a couple of two reasons: first, convexity is not a biholomorphic invariant, and second, *there were not many explicit examples available at the time.* (The plot thickens on this problem: there is a domain, namely the symmetrized bidisk, in $\mathbb{C}^2$ for which the two distances agree, yet this domain is not biholomorphically equivalent to a convex domain. See [8] for a summary of these results.)

While we cannot remedy the problem of a lack of examples in the past, we can attempt to add to the current selection of explicit examples. The excellent book by Kobayashi [9] presents many remarkable theorems applicable in the generality of complex spaces (as the title suggests) about the above invariant metrics (and applications thereof), yet curiously there do not seem to be any explicit examples of the Carathéodory distance for a complex space *with a singularity.* Perhaps the simplest complex space with a singularity is *Neile's Semicubical Parabola*[1] which is the variety contained in the bidisk given by $\{(z, w) \in \mathbb{D}^2 : z^2 = w^3\}$. We shall call it the *Neile Parabola* for short. In their recent follow-up [8] to their book [7], M. Jarnicki and P. Pflug pose the following question. Is there an effective formula for the Carathéodory distance on the Neile parabola? In this paper, we give an answer to this question (see theorem 4.1). In addition, we compute the infinitesimal Carathéodory pseudo-distance for the Neile parabola (see theorem 4.2).

2. A Mixed Carathéodory-Pick Problem

This problem also has a connection with interpolation problems on the disk for bounded analytic functions. Given n points in the unit disk z_i and n target values w_i also in the unit disk, the well-known theorem of G. Pick [12] says exactly when there exists a holomorphic $f : \mathbb{D} \to \mathbb{D}$ satisfying $f(z_i) = w_i$ (this problem was studied independently by Nevanlinna [11]). In fact the Schwarz-Pick lemma is just the version of this for two points: z_1, z_2 can be interpolated to w_1, w_2 if and only if

$$\left|\frac{w_1 - w_2}{1 - \bar{w}_1 w_2}\right| \leq \left|\frac{z_1 - z_2}{1 - \bar{z}_1 z_2}\right|$$

Similarly, given n complex numbers $a_0, a_1, \ldots, a_{n-1}$ a well-known theorem of Carathéodory and Fejér [3] says when there exists a holomorphic function $f : \mathbb{D} \to \mathbb{D}$ with $a_0, a_1, \ldots, a_{n-1}$ as the first n Taylor

[1]Named after William Neile, a student of John Wallis, it was the first algebraic curve to have its arc length computed [14]. Of course, he was computing the arc length of the real curve $y^2 = x^3$.

coefficients of f. For $n = 2$, this is given again by the (infinitesimal) Schwarz-Pick lemma: a_0 and a_1 can be the first two Taylor coefficients exactly when

$$\frac{|a_1|}{1 - |a_0|^2} \leq 1$$

The first kind of interpolation problem above is called Nevanlinna-Pick interpolation and the second is called Carathéodory-Fejér interpolation. More modern proofs of these theorems, using ideas from operator theory like the commutant lifting theorem of Sz.-Nagy and Foias and reproducing kernel Hilbert spaces, make it possible to study so-called *mixed Carathéodory-Pick problems* wherein the idea is to specify several Taylor coefficients at several points in the disk and determine whether there exists a holomorphic function from the disk to the disk with those properties. However, a restriction imposed in all of the usual mixed Carathéodory-Pick problems is that the Taylor coefficients must be specified in a sequence. For example, these problems do not address an interpolation problem of the following form: given $z_1, z_2, z_3, w_1, w_2 \in \mathbb{D}$, when is there a holomorphic function $f : \mathbb{D} \to \mathbb{D}$ satisfying the following?

$$\begin{aligned} f(z_1) &= w_1 \\ f(z_2) &= w_2 \\ f'(z_3) &= 0 \end{aligned} \tag{2.1}$$

In fact, as we shall see, solving the problem (2.1) amounts to computing the Carathéodory distance for the Neile parabola. See proposition 4.5 for the exact statement of our result. It should be mentioned that this question can be reformulated as the following question involving a traditional mixed Carathédory-Pick problem: does there exist a $w_3 \in \mathbb{D}$ so that the mixed Carathódory-Pick problem given by (2.1) and the additional condition $f(z_3) = w_3$ has a solution? This reformulation does not, however, reduce of the difficulty of the problem.

3. Extension of Bounded Holomorphic Functions on the Neile Parabola

The following result is a special case of the work of H. Cartan on Stein Varieties (see [5] page 99). (In fact, we are stating it in almost as little generality as possible.)

Theorem 3.1 (Cartan). *Every holomorphic function on a subvariety V of $\mathbb{D}^2$ is the restriction of a holomorphic function on all of $\mathbb{D}^2$.*

A vast improvement on this theorem (again stated in simple terms) was given by P.L. Polyakov and G.M. Khenkin [13]. They proved using the methods of integral formulas that any subvariety V of $\mathbb{D}^2$ satisfying a certain transversality condition has the property that any bounded holomorphic function on V can be extended to a bounded holomorphic function on all of $\mathbb{D}^2$. In fact, there is a bounded linear operator $T : H^\infty(V) \to H^\infty(\mathbb{D}^2)$ with $Tf\mid_V = f$, so that there is some constant C such that for any $f \in H^\infty(V)$

$$||Tf||_\infty \leq C||f||_\infty \tag{3.2}$$

The previously mentioned "transversality condition" applies to the Neile parabola, and therefore any bounded holomorphic function on M can be extended to a bounded holomorphic function on the bidisk.

Related to these ideas is a paper of J. Agler and J.E. McCarthy [2], which gives a description of varieties in the bidisk with the property that bounded holomorphic functions can be extended to the bidisk without increasing their H^∞ norm. The Neile parabola is not such a variety as their results show. This can be seen relatively easily from the fact that the Carathéodory pseudo-distance on the Neile parabola is not the restriction of the Carathéodory pseudo-distance on the bidisk. Meaning, there is some holomorphic function from M to $\mathbb{D}$ which separates two points of M farther than a function from the bidisk to the disk could. Hence, such a function could not be extended to the bidisk without increasing its norm.

This suggests that extremal functions on the Neile parabola for the Carathéodory pseudo-distance might be good candidates for functions which extend "badly" to the bidisk. Indeed, this allows us to give a lower bound of $5/4$ on the constant C in (3.2) for the Neile Parabola. In addition to this we present a simple proof using Agler's Nevanlinna-Pick interpolation theorem for the bidisk that any bounded holomorphic function on the Neile parabola can be extended to a bounded holomorphic function on the bidisk with norm increasing by at most a factor of $\sqrt{2}$ if the function vanishes at the origin and by a factor of $2\sqrt{2}+1$ otherwise. This does not exactly reprove Polyakov and Khenkin's result in our context, since we are not claiming the extension can be given by a linear operator. Nevertheless, it is certainly relevant to their result, is much easier to prove, and provides an explicit bound (see theorem 4.7).

4. Definitions and Statements of Results

Let us define several important notions for this paper. We shall use $\mathcal{O}(X,Y)$ to denote the set of holomorphic maps from X to Y and $\mathcal{O}(X)$ to denote the set of holomorphic functions from X to $\mathbb{C}$.

- The *pseudo-hyperbolic distance* on the unit disk $\mathbb{D} \subset \mathbb{C}$ is defined to be
$$m(a,b) = \left|\frac{a-b}{1-\bar{a}b}\right|$$
The *Poincaré distance* on $\mathbb{D}$ is given by $\rho = \tanh^{-1} m$.
- The *Poincaré metric* on the disk, which we shall also denote by ρ, is defined to be
$$\rho(z;v) = \frac{|v|}{1-|z|^2}$$
for $z \in \mathbb{D}$ and $v \in \mathbb{C}$.
- Given a complex analytic set X (or a complex manifold or a domain in $\mathbb{C}^n$) the *Carathéodory pseudo-distance* on X is denoted by c_X and is defined by
$$c_X(x,y) := \sup\{\rho(f(x),f(y)) : f \in \mathcal{O}(X,\mathbb{D})\}$$
If we replace ρ above with m, we get what Jarnicki and Pflug call the *Möbius pseudo-distance*:
$$c_X^*(x,y) := \sup\{m(f(x),f(y)) : f \in \mathcal{O}(X,\mathbb{D})\}$$
Due to the simple formula for m and the relation $c_X = \tanh^{-1} c_X^*$, the Möbius pseudo-distance is more computationally useful for our purposes, and therefore will be used exclusively in all proofs.
- Again, for a complex space X, the *Carathéodory pseudo-metric* E_X is defined to be
$$E_X(x;v) = \sup\{\rho(f(x);df_x(v)) : f \in \mathcal{O}(X,\mathbb{D})\}$$
for $x \in X$ and $v \in T_xX$, the tangent space of X at x. The Carathéodory pseudo-metric will often be referred to as the *infinitesimal Carathéodory pseudo-distance*.
- Finally, the *Lempert function* for X, as above, is denoted $\tilde{k}_X$ and is defined by
$$\tilde{k}_X(x,y) = \inf\{\rho(a,b) : \exists f \in \mathcal{O}(\mathbb{D},X) \text{ with } f(a)=x, f(b)=y\}$$
where $\tilde{k}_X$ is defined to equal ∞ if the above set over which the infimum is taken is empty. The *Kobayashi pseudo-distance* k_X is then defined to be largest pseudo-distance bounded by $\tilde{k}_X$.

For more information on these definitions we refer the reader to [6], [7], [8], and [9].

In [8] on page 8, Jarnicki and Pflug pose the following question. Let $M = \{(z,w) \in \mathbb{D}^2 : z^2 = w^3\}$ be the *Neile parabola*. The set M is a one-dimensional connected analytic variety in $\mathbb{D}^2$ with a singularity at $(0,0)$. Furthermore, M has a bijective holomorphic parameterization $p : \mathbb{D} \to M$ given by

$$p(\lambda) := (\lambda^3, \lambda^2)$$

The function $q := p^{-1}$ is continuous on M, holomorphic on $M \setminus \{(0,0)\}$, and can be given by $q(z,w) = z/w$ when $(z,w) \neq (0,0)$ (and $q(0,0) = 0$). For the benefit of those readers unfamiliar with holomorphic functions on a variety with a singularity, we include a discussion of these ideas in the concrete context of the Neile parabola in section 5. It is known that the Kobayashi pseudo-distance k_M and the Lempert function $\tilde{k}_M$ for M are as simple as possible:

$$k_M = \tilde{k}_M((a,b),(z,w)) = \rho(q(a,b), q(z,w))$$

(For the sake of completeness, we include a proof of this in section 5.) On the other hand, in [8] the authors lament that despite M being so simple, an effective formula for the Carathéodory pseudo-distance c_M is not known. We propose the following as an effective formula for c_M.

First, for $a \in \mathbb{D}$ let $\phi_a : \mathbb{D} \to \mathbb{D}$ denote the automorphism of the disk given by

$$\phi_a(z) = \frac{a - z}{1 - \bar{a}z}$$

Theorem 4.1 (Carathéodory Pseudo-Distance formula)**.** *Given nonzero $\lambda, \delta \in \mathbb{D}$ let*

$$\alpha_0 = \frac{1}{2}\left(\frac{1}{\bar{\lambda}} + \lambda + \frac{1}{\bar{\delta}} + \delta\right)$$

then

$$c_M(p(\lambda), p(\delta)) = \begin{cases} \rho(\lambda^2, \delta^2) & \text{if } |\alpha_0| \geq 1 \\ \rho(\lambda^2 \phi_{\alpha_0}(\lambda), \delta^2 \phi_{\alpha_0}(\delta)) & \text{if } |\alpha_0| < 1 \end{cases}$$

Also, $c_M(p(0), p(\lambda)) = \rho(0, \lambda^2) = \tanh^{-1} |\lambda|^2$.

In particular, it should be noted that if λ and δ have an acute angle between them (i.e. $\mathrm{Re}\lambda\bar{\delta} > 0$), then $|\alpha_0| > 1$, and the first formula above gives the distance between $p(\lambda)$ and $p(\delta)$.

In section 6 we shall reduce the above problem to a maximization problem on the closed unit disk, and in the following section we solve the maximization problem to yield theorem 4.1. In addition, a slightly nicer form of the above formula will be presented.

As will be explained in section 5, the tangent spaces of M can be identified with subspaces of the tangent spaces of $\mathbb{D}^2$. In particular, for $x = (a,b) \neq (0,0)$, $T_x M$ is simply the span of the vector $(3a, 2b)$, while the tangent space at the origin of M is two dimensional and therefore equal to all of $\mathbb{C}^2 = T_{(0,0)}\mathbb{D}^2$. We can now present our formula for the Carathéodory pseudo-metric of M (this is proved in section 8).

Theorem 4.2 (Carathédory Pseudo-metric formula)**.** *For $v = (v_1, v_2) \in \mathbb{C}^2$, we have*

$$E_M((0,0);v) = \begin{cases} |v_2| & \text{if } |v_2| \geq 2|v_1| \\ \frac{4|v_1|^2+|v_2|^2}{4|v_1|} & \text{if } |v_2| < 2|v_1| \end{cases} \tag{4.3}$$

and for $(a,b) \in M$ nonzero and $z \in \mathbb{C}$ we have

$$E_M((a,b); z(3a,2b)) = \frac{2|b|}{1-|b|^2}|z| \tag{4.4}$$

As mentioned in section 2, as a direct consequence of preceding formulas, we can prove the following atypical mixed Carathéodory-Pick interpolation result (see section 9).

Proposition 4.5 (Mixed Interpolation Problem)**.** *First, given distinct $z_1, z_2, z_3 \in \mathbb{D}$ and $w_1, w_2 \in \mathbb{D}$, there exists $f \in \mathcal{O}(\mathbb{D}, \mathbb{D})$ with*

$$\begin{aligned} f(z_i) &= w_i \text{ for } i = 1,2 \\ f'(z_3) &= 0 \end{aligned}$$

if and only if

$$\rho(w_1, w_2) \leq c_M(p(\phi_{z_3}(z_1)), p(\phi_{z_3}(z_2))) \tag{4.6}$$

Moreover, if the problem is extremal (i.e. if there is equality in (4.6)), then the solution is unique and is a Blaschke product of order two or three.

Finally, in section 10 we prove the following result on extending bounded holomorphic functions from the Neile parabola to the bidisk.

Theorem 4.7 (Bounded Analytic Extension)**.** *For any $f \in \mathcal{O}(M, \mathbb{D})$ with $f(0,0) = 0$, there exists an extension of f to a function in $\mathcal{O}(\mathbb{D}^2, \sqrt{2}\mathbb{D})$. If $f(0,0) \neq 0$, then f can be extended to $\mathcal{O}(\mathbb{D}^2, (2\sqrt{2}+1)\mathbb{D})$. In addition, there exists a function $g \in \mathcal{O}(M, \mathbb{D})$ which cannot be extended to a function in $\mathcal{O}(\mathbb{D}^2, r\mathbb{D})$ for $r < 5/4$.*

5. Complex Analysis on the Neile Parabola

In this section we discuss how to do complex analysis on a variety with a singularity in the concrete setting of the Neile parabola. This section is adapted from [8] and [4] (see pages 18-20 and the chapter on tangent spaces) and no results in this section are by any means new. A function f on M is defined to be holomorphic if at each point $x \in M$, there is a holomorphic function F on a neighborhood U of x in the bidisk which agrees with f on $U \cap M$. Fortunately, we can give a more concrete description of the set of holomorphic functions on M.

Given $f \in \mathcal{O}(M)$, the function $h := f \circ p$ is an element of $\mathcal{O}(\mathbb{D})$ satisfying $h'(0) = 0$. The reason for this is given an extension, F, of f holomorphic on a neighborhood of $(0,0)$ in $\mathbb{D}^2$, $h = F \circ p$ is holomorphic on a neighborhood of 0 in $\mathbb{D}$. Hence, the derivative $h'(\lambda) = dF_{p(\lambda)}(3\lambda^2, 2\lambda)$ and so $h'(0) = 0$.

Conversely, suppose $h \in \mathcal{O}(\mathbb{D})$ satisfies $h'(0) = 0$. Then, $f := h \circ q$ is holomorphic on $M \setminus \{(0,0)\}$ because $F(z,w) = h(z/w)$ is holomorphic on the set $\{(z,w) \in \mathbb{D}^2 : |z| < |w|\}$ which is an open neighborhood of of $M \setminus \{(0,0)\}$. To prove f is holomorphic at $(0,0)$, observe first of all that h can be written as an absolutely convergent power series $h(\lambda) = a_0 + a_2\lambda^2 + a_3\lambda^3 + \cdots$ in some (or any) closed disk contained in $\mathbb{D}$ and centered at the origin (of radius say r). Then, for (z,w) with $|z| < 1, |w| < r^3$,

$$F(z,w) := a_0 + a_2 w + a_3 z + a_4 w^2 + a_5 zw + a_6 w^3 + \cdots$$

converges absolutely and extends f (where we are choosing to extend $(z/w)^k$ to a monomial of the form zw^m or w^m—i.e. we want the power of w to be as large as possible).

This establishes the correspondence between $\mathcal{O}(M)$ and the functions in $\mathcal{O}(\mathbb{D})$ whose derivatives vanish at 0.

Now, we can prove the formula for the Kobayashi pseudo-distance and the Lempert function:

Proposition 5.1.

$$k_M(p(\lambda), p(\delta)) = \tilde{k}_M(p(\lambda), p(\delta)) = \rho(\lambda, \delta)$$

Proof. First, $\tilde{k}_M(p(\lambda), p(\delta)) \leq \rho(\lambda, \delta)$ because p is holomorphic. Second, if $f = (f_1, f_2) \in \mathcal{O}(\mathbb{D}, M)$ then $g := q \circ f$ is holomorphic for the following reasons. At any point $a \in \mathbb{D}$ where $f(a) \neq (0,0)$ it is clear that g is holomorphic. If $f(a) = 0$, then, since $f_1^2 = f_2^3$, it follows that for some positive integer k, f_1 has a zero of order $3k$ and f_2 has a zero of order $2k$ at z. Hence, $g = f_1/f_2$ is holomorphic in a neighborhood of a (as the singularity at a is removable). Therefore,

if $f(a) = p(\lambda)$ and $f(b) = p(\delta)$, then $\rho(\lambda, \delta) = \rho(g(a), g(b)) \leq \rho(a, b)$ by the Schwarz lemma. Taking the infimum over all such a, b we get $\rho(\lambda, \delta) \leq \tilde{k}_M(p(\lambda), p(\delta))$. That proves the formula for the Lempert function. The Kobayashi pseudo-distance is equal to the Lempert function because the Lempert function is already a pseudo-distance. $\square$

Next, we discuss the complex tangent spaces of M. We can define T_xM as a subset of $T_x\mathbb{D}^2 \cong \mathbb{C}^2$ in the following way. Given $v \in \mathbb{C}^2$, $v \in T_xM$ if and only if $dG_xv = 0$ for every holomorphic function G in a neighborhood U (in $\mathbb{D}^2$) of x with G identically zero restricted to $U \cap M$. Notice that this definition is designed to make it easy to define the differential of a function $g \in \mathcal{O}(M)$.

If $x = p(\lambda) = (a, b) \neq (0, 0)$ then T_xM is the span of the vector $(3a, 2b)$, because for G as before the function $f := G \circ p$ is identically zero and so $0 = f'(\lambda) = dG_x(3\lambda^2, 2\lambda)$. Hence, $dG_x(3a, 2b) = 0$. On the other hand, $h(z, w) = z^2 - w^3$ vanishes on M and $dh_xv = 0$ if and only if v is a multiple of $(3a, 2b)$.

At the origin $x = (0, 0)$, we have $T_xM = \mathbb{C}^2$, because if G is again as above, then $dG_{(0,0)} = (0, 0)$. This is because the partial derivatives of G at $(0, 0)$ are the coefficients of λ^3 and λ^2 in the identically zero power series for $G(\lambda^3, \lambda^2)$.

6. Reduction of the Problem

As mentioned earlier, we shall compute a formula for c_M^* (which of course gives a formula for c_M).

Because of preceding section, we immediately have

$$c_M^*(p(\lambda), p(\delta)) = \sup\{m(h(\lambda), h(\delta)) : h \in \mathcal{O}(\mathbb{D}, \mathbb{D}), h'(0) = 0\} \tag{6.1}$$

As m is invariant under automorphisms of the disk, we may assume $h(0) = 0$ by applying appropriate automorphisms of the disk, since the condition $h'(0) = 0$ is preserved by (post) composition. Then, by the Schwarz lemma, h may be written as $h(z) = z^2 f(z)$ for some $f \in \mathcal{O}(\mathbb{D}, \overline{\mathbb{D}})$. At this stage it is clear that $c_M^*(p(0), p(\lambda)) = |\lambda|^2$. As f varies over all of $\mathcal{O}(\mathbb{D}, \overline{\mathbb{D}})$, the set of pairs $(f(\lambda), f(\delta))$ is just the set of all (a, b) satisfying $m(a, b) \leq m(\lambda, \delta)$. Hence,

$$c_M^*(p(\lambda), p(\delta)) = \sup\{m(\lambda^2 a, \delta^2 b) : m(a, b) \leq m(\lambda, \delta)\}$$

Since $m(\lambda^2 a, \delta^2 b)$ is the modulus of a holomorphic function in a, the above supremum may be taken over all (a, b) with $m(a, b) = m(\lambda, \delta)$, by the maximum principle. We may safely multiply both a and b a unimodular constant and leave $m(\lambda^2 a, \delta^2 b)$ unchanged. Thus, we can assume there is some $\alpha \in \mathbb{D}$ such that $a = \phi_\alpha(\lambda)$ and $b = \phi_\alpha(\delta)$.

Keeping λ and δ fixed from now on, we define a continuous function, smooth except possibly where it is zero, $F : \overline{\mathbb{D}} \to [0,1)$ by

$$F(\alpha) := m(\lambda^2\phi_\alpha(\lambda), \delta^2\phi_\alpha(\delta)) \tag{6.2}$$

A couple of things to notice about F are $F(\alpha) < m(\lambda,\delta)$ for all $\alpha \in \overline{\mathbb{D}}$ and $F(\alpha) = m(\lambda^2,\delta^2)$ for all α with $|\alpha| = 1$. By the preceding discussion we may conclude:

Proposition 6.3.

$$c^*_M(p(\lambda), p(\delta)) = \sup_{\alpha\in\overline{\mathbb{D}}} F(\alpha) = \sup_{\alpha\in\overline{\mathbb{D}}} m(\lambda^2\phi_\alpha(\lambda), \delta^2\phi_\alpha(\delta))$$

In particular, the supremum in (6.1) is attained by some function of the form $h(\zeta) = \zeta^2\phi_\alpha(\zeta)$ where $\alpha \in \overline{\mathbb{D}}$. Moreover, if h attains the supremum in (6.1) and $h(0) = 0$, then h is of the same form (i.e. $h = \zeta^2\phi_\alpha$) up to multiplication by a unimodular constant. As we shall see later, either the supremum will be obtained with a unique $\alpha \in \mathbb{D}$ or with any $\alpha \in \partial\mathbb{D}$.

Some computations yield a couple of useful formulas for F:

Claim 6.4.

$$\begin{aligned} F(\alpha) &= \\ &= m(\lambda,\delta)\left|\frac{(\lambda+\delta)(\alpha+\lambda\delta\bar{\alpha}-\lambda-\delta)+\lambda\delta(1-|\alpha|^2)}{(1+\lambda\bar{\delta})(1+\lambda\bar{\delta}-\bar{\alpha}\lambda-\alpha\bar{\delta})-\lambda\bar{\delta}(1-|\alpha|^2)}\right| &(6.5)\\ &= m(\lambda,\delta)\left|\frac{1-(\bar{\alpha}-\bar{\alpha}_0-\bar{\beta}_2)(\alpha-\alpha_0+\beta_2)}{1-(\bar{\alpha}-\bar{\alpha}_0-\bar{\beta}_1)(\alpha-\alpha_0+\beta_1)}\right| &(6.6)\end{aligned}$$

where

$$\begin{aligned} \alpha_0 &:= \frac{1}{2}\left(\frac{1}{\bar{\lambda}}+\lambda+\frac{1}{\bar{\delta}}+\delta\right), \\ \beta_1 &:= \frac{1}{2}\left(\frac{1}{\bar{\lambda}}-\lambda-\frac{1}{\bar{\delta}}+\delta\right), \text{ and} \\ \beta_2 &:= \frac{1}{2}\left(\frac{1}{\bar{\lambda}}-\lambda+\frac{1}{\bar{\delta}}-\delta\right)\end{aligned}$$

Proof of Claim: We start from equation (6.2). Observe that

$$
\begin{aligned}
F(\alpha) &= \left|\frac{\lambda^2\frac{\alpha-\lambda}{1-\bar\alpha\lambda}-\delta^2\frac{\alpha-\delta}{1-\bar\alpha\delta}}{1-\lambda^2\bar\delta^2\frac{\alpha-\lambda}{1-\bar\alpha\lambda}\frac{\bar\alpha-\bar\delta}{1-\alpha\bar\delta}}\right| \\
&= \left|\frac{\lambda^2(\alpha-\lambda)(1-\bar\alpha\delta)-\delta^2(\alpha-\delta)(1-\bar\alpha\lambda)}{(1-\bar\alpha\lambda)(1-\alpha\bar\delta)-\lambda^2\bar\delta^2(\alpha-\lambda)(\bar\alpha-\bar\delta)}\right| \\
&= \left|\frac{\alpha(\lambda^2-\delta^2)-(\lambda^3-\delta^3)-|\alpha|^2\lambda\delta(\lambda-\delta)+\lambda\delta(\lambda^2-\delta^2)\bar\alpha}{1-\lambda^3\bar\delta^3-\bar\alpha\lambda(1-\lambda^2\bar\delta^2)-\alpha\bar\delta(1-\lambda^2\bar\delta^2)+|\alpha|^2\lambda\bar\delta(1-\lambda\bar\delta)}\right| \\
(6.7) = \quad & m(\lambda,\delta)\left|\frac{\alpha(\lambda+\delta)-(\lambda^2+\lambda\delta+\delta^2)-|\alpha|^2\lambda\delta+\lambda\delta(\lambda+\delta)\bar\alpha}{1+\lambda\bar\delta+\lambda^2\bar\delta^2-\bar\alpha\lambda(1+\lambda\bar\delta)-\alpha\bar\delta(1+\lambda\bar\delta)+|\alpha|^2\lambda\bar\delta}\right| \\
&= m(\lambda,\delta)\left|\frac{\alpha(\lambda+\delta)+\lambda\delta(\lambda+\delta)\bar\alpha-(\lambda+\delta)^2+\lambda\delta(1-|\alpha|^2)}{(1+\lambda\bar\delta)^2-\bar\alpha\lambda(1+\lambda\bar\delta)-\alpha\bar\delta(1+\lambda\bar\delta)-(1-|\alpha|^2)\lambda\bar\delta}\right|
\end{aligned}
$$

and from here it is easy to get (6.5).

Secondly, to prove (6.6), we start from (6.7):

$$
\begin{aligned}
F(\alpha) &= m(\lambda,\delta)\left|\frac{\lambda\delta-(\lambda\delta\bar\alpha-(\lambda+\delta))(\alpha-(\lambda+\delta))}{\lambda\bar\delta-(\bar\alpha\lambda-(1+\lambda\bar\delta))(\alpha\bar\delta-(1+\lambda\bar\delta))}\right| \\
&= m(\lambda,\delta)\left|\frac{1-(\bar\alpha-(1/\delta+1/\lambda))(\alpha-(\lambda+\delta))}{1-(\bar\alpha-(1/\lambda+\bar\delta))(\alpha-(1/\bar\delta+\lambda))}\right|
\end{aligned}
$$

and this equals (6.6) because of the identities:

$$
\begin{aligned}
\bar\alpha_0+\bar\beta_2 &= \frac{1}{\lambda}+\frac{1}{\delta} \\
\alpha_0-\beta_2 &= \lambda+\delta \\
\bar\alpha_0+\bar\beta_1 &= \frac{1}{\lambda}+\bar\delta \\
\alpha_0-\beta_1 &= \lambda+\frac{1}{\bar\delta}
\end{aligned}
$$

□

7. Proof of Theorem 4.1

In this section, we prove two things. First, F has no local maximum in $\mathbb{D}$ except possibly α_0. Second, when $|\alpha_0|<1$, $F(\alpha)\le F(\alpha_0)$ for all α with $|\alpha|=1$. These two claims yield theorem 4.1.

Lemma 7.1. *The function F has no local maximum in $\mathbb{D}$ except possibly at α_0.*

Proof. Using the formula 6.6, it suffices to prove the function given by

$$G(z) = \left|\frac{1-(\bar{z}-\bar{\beta}_2)(z+\beta_2)}{1-(\bar{z}-\bar{\beta}_1)(z+\beta_1)}\right|^2 \tag{7.2}$$

has no local max for $|z+\alpha_0|<1$ except possibly at $z=0$. Yet more computations show that G can be written as G_2/G_1 where

$$G_k(z) = 1+2|\beta_k|^2 - 2|z|^2 + |z^2-\beta_k^2|^2 \tag{7.3}$$

for $k=1,2$ (recall that β_1 and β_2 were defined in the previous section). Indeed, using the identity $|1-a\bar{b}|^2 - |a+b|^2 = (1-|a|^2)(1-|b|^2)$ we have

$$\begin{aligned}
&|1-(\bar{z}-\bar{\beta}_k)(z+\beta_k)|^2 \\
&= 4|\beta_k|^2 + (1-|z-\beta_k|^2)(1-|z+\beta_k|^2) \\
&= 1+4|\beta_k|^2 - |z-\beta_k|^2 - |z+\beta_k|^2 + |z^2-\beta_k^2|^2 \\
&= 1+2|\beta_k|^2 - 2|z|^2 + |z^2-\beta_k^2|^2
\end{aligned}$$

as desired.

Throughout the following, suppose z is a local maximum satisfying $0<|z+\alpha_0|<1$. In particular, this implies several things:

- $0<G(z)<1$,
- z is a critical point for G,
- $\Delta \log G(z) \leq 0$, and
- $\det \mathrm{Hess}(\log G) \geq 0$ at z.

Here Hess denotes the matrix of second partial derivatives. We will prove that all of these conditions cannot be satisfied.

Let's compute all of the derivatives of G_1 and G_2 up to second order. Luckily we can examine G_1 and G_2 simultaneously. Writing $z=x+iy$ we have

$$\begin{array}{rcl}
\partial_z G_k &=& -2\bar{z} + 2z(\bar{z}^2 - \bar{\beta}_k{}^2) \\
\partial_x G_k &=& -4x + 4\mathrm{Re}[z(\bar{z}^2 - \bar{\beta}_k{}^2)] \\
\partial_y G_k &=& -4y - 4\mathrm{Im}[z(\bar{z}^2 - \bar{\beta}_k{}^2)] \\
\partial^2_{xx} G_k &=& -4 + 4|z|^2 + 8x^2 - 4\mathrm{Re}\beta_k^2 \\
\partial^2_{yy} G_k &=& -4 + 4|z|^2 + 8y^2 + 4\mathrm{Re}\beta_k^2 \\
\partial^2_{xy} G_k &=& 8xy - 4\mathrm{Im}\beta_k^2
\end{array}$$

Since z is a critical point for G, we have $G_1\partial_z G_2 - G_2 \partial_z G_1 = 0$ at z. Neither G_1 nor G_2 vanish at z, and as a result if $\partial_z G_1 = 0$ then

$\partial_z G_2 = 0$. But, $\partial_z G_1$ and $\partial_z G_2$ vanish simultaneously only at 0:

$$\partial_z G_k = -2\bar{z} + 2z(\bar{z}^2 - \bar{\beta}_k{}^2) = 0$$

for $k = 1, 2$ implies $\bar{z}(\beta_1^2 - \beta_2^2) = 0$, which can only happen if $z = 0$ (because $\beta_1^2 - \beta_2^2 = -(1 - |\lambda|^2)(1 - |\delta|^2)/(\bar{\lambda}\bar{\delta}) \neq 0$). Therefore, at z

$$\frac{G_2}{G_1} = \frac{\partial_z G_2}{\partial_z G_1}, \qquad \frac{\partial_x G_1}{G_1} = \frac{\partial_x G_2}{G_2}, \quad \text{and} \quad \frac{\partial_y G_1}{G_1} = \frac{\partial_y G_2}{G_2} \tag{7.4}$$

A fact derived from the first of these equations is

$$\left(\frac{\bar{\beta}_1{}^2}{G_1} - \frac{\bar{\beta}_2{}^2}{G_2}\right) z^2 = |z|^2(1 - |z|^2)\left(\frac{1}{G_2} - \frac{1}{G_1}\right) \tag{7.5}$$

and in particular the expression on the left is real.

Using the last two equations in (7.4), we can see that at the critical point z the following equations hold

$$\begin{aligned}
\partial_{xx}^2 \log G &= \frac{\partial_{xx} G_2}{G_2} - \frac{\partial_{xx}^2 G_1}{G_1} \\
&= (-4 + 4|z|^2 + 8x^2)\left(\frac{1}{G_2} - \frac{1}{G_1}\right) + 4\mathrm{Re}\left(\frac{\bar{\beta}_1{}^2}{G_1} - \frac{\bar{\beta}_2{}^2}{G_2}\right) \\
&= -4[(1 - |z|^2)(1 - \mathrm{Re}(z^2/|z|^2)) - 2x^2]\left(\frac{1}{G_2} - \frac{1}{G_1}\right)
\end{aligned}$$

where the last equality follows from (7.5). Similarly,

$$\begin{aligned}
\partial_{yy}^2 \log G &= -4[(1 - |z|^2)(1 + \mathrm{Re}(z^2/|z|^2)) - 2y^2]\left(\frac{1}{G_2} - \frac{1}{G_1}\right) \\
\partial_{xy}^2 \log G &= 4[2xy + (1 - |z|^2)\mathrm{Im}(z^2/|z|^2)]\left(\frac{1}{G_2} - \frac{1}{G_1}\right)
\end{aligned}$$

Therefore,

$$\Delta \log G = -8(1 - 3|z|^2)\left(\frac{1}{G_2} - \frac{1}{G_1}\right)$$

and as this must be less than or equal to zero at z, we see that $|z|^2 \leq 1/3$.

Finally, we can show that $\det \mathrm{Hess}(\log G) < 0$, contradicting the fact that z is assumed to be a local maximum. The determinant of the Hessian of the logarithm of G (with the positive factor $16(1/G_2 - 1/G_1)^2$

omitted) is

$$\begin{aligned}&(1-|z|^2)^2(1-(\mathrm{Re}(z^2/|z|^2))^2)+4x^2y^2-2|z|^2(1-|z|^2)\\&+2(y^2-x^2)(1-|z|^2)\mathrm{Re}(z^2/|z|^2)\\&-4x^2y^2-4xy(1-|z|^2)\mathrm{Im}(z^2/|z|^2)-(1-|z|^2)^2(\mathrm{Im}(z^2/|z|^2))^2\end{aligned}$$

Canceling the positive factor $(1-|z|^2)$ and simplifying, we get

$$-4|z|^2<0$$

as desired.

□

Lemma 7.6. *If $|\alpha_0|<1$, then $F(\alpha)\le F(\alpha_0)$ for all α with $|\alpha|=1$.*

Proof. As mentioned earlier, on the boundary of $\overline{\mathbb{D}}$, F is constant and equal to $m(\lambda^2,\delta^2)$. From equation (6.5) it suffices to prove the inequality

$$\left|\frac{\lambda+\delta}{1+\bar\lambda\delta}\right|^2\le\left|\frac{(\lambda+\delta)(\alpha_0+\lambda\delta\bar\alpha_0-\lambda-\delta)+\lambda\delta(1-|\alpha_0|^2)}{(1+\lambda\bar\delta)(1+\lambda\bar\delta-\bar\alpha_0\lambda-\alpha_0\bar\delta)-\lambda\bar\delta(1-|\alpha_0|^2)}\right|^2$$

Assuming the left hand side above is nonzero (which we can), it suffices to prove

$$\begin{aligned}&\left|(\alpha_0+\lambda\delta\bar\alpha_0-\lambda-\delta)+\lambda\delta\frac{(1-|\alpha_0|^2)}{\lambda+\delta}\right|^2\\&-\left|(1+\lambda\bar\delta-\bar\alpha_0\lambda-\alpha_0\bar\delta)-\lambda\bar\delta\frac{(1-|\alpha_0|^2)}{1+\lambda\bar\delta}\right|^2\ge0\end{aligned}\tag{7.7}$$

If we think of the left hand side as $|A+B|^2-|C+D|^2$=$|A|^2-|C|^2+2\mathrm{Re}(A\bar B-C\bar D)+|B|^2-|D|^2$, then first of all $|A|^2-|C|^2$ equals

$$|\alpha_0+\lambda\delta\bar\alpha_0-\lambda-\delta|^2-|1+\lambda\bar\delta-\bar\alpha_0\lambda-\alpha_0\bar\delta|^2=-(1-|\alpha_0|^2)(1-|\lambda|^2)(1-|\delta|^2)$$

and using the identities

$$\begin{aligned}\alpha_0+\lambda\delta\bar\alpha_0-(\lambda+\delta)&=\frac{\bar\lambda+\bar\delta}{2\bar\lambda\bar\delta}(1+|\lambda\delta|^2)\\1+\lambda\bar\delta-\bar\alpha_0\lambda-\alpha_0\bar\delta&=-\frac{1+\bar\lambda\delta}{2\bar\lambda\delta}(|\lambda|^2+|\delta|^2)\end{aligned}$$

we get $2\mathrm{Re}(A\bar B-C\bar D)=(1-|\alpha_0^2)(1-|\lambda|^2)(1-|\delta|^2)$.

Also, using the identity

$$|1+a\bar b|^2-|a+b|^2=(1-|a|^2)(1-|b|^2)\tag{7.8}$$

we see that $|B|^2 - |D|^2$ equals

$$|\lambda\delta|^2(1-|\alpha_0|^2)^2\frac{(1-|\lambda|^2)(1-|\delta|^2)}{|\lambda+\delta|^2|1+\lambda\bar{\delta}|^2}$$

Summing this all up, we see that proving (7.7) amounts to showing

$$|\lambda\delta|^2(1-|\alpha_0|^2)^2\frac{(1-|\lambda|^2)(1-|\delta|^2)}{|\lambda+\delta|^2|1+\lambda\bar{\delta}|^2} \geq 0$$

which is certainly true.

□

This concludes the proof of theorem (4.1). As promised, a slightly nicer formula for $c^*_M(p(\lambda), p(\delta))$ is

Proposition 7.9. *If $\lambda, \delta \in \mathbb{D}$ are nonzero, then*

$$c^*_M(p(\lambda), p(\delta)) = \begin{cases} m(\lambda^2, \delta^2) & \text{if } |\alpha_0| \geq 1 \\ m(\lambda,\delta)\frac{1+|\beta_2|^2}{1+|\beta_1|^2} & \text{if } |\alpha_0| < 1 \end{cases}$$

This follows from the formula (6.6) for F.

8. The Infinitesimal Carathéodory Pseudo-distance

In this section we prove theorem 4.2, our formula for the Carathéodory pseudo-metric.

The Carathéodory pseudo-metric at the origin and a vector $v = (v_1, v_2) \in \mathbb{C}^2$ is

$$E_M((0,0); v) = \sup\{|dg_{(0,0)}v| : g \in \mathcal{O}(M, \mathbb{D}) \text{ and } g(0,0) = 0\}$$

Any g as above satisfies $g(\lambda^3, \lambda^2) = \lambda^2 f(\lambda)$ for some $f \in \mathcal{O}(\mathbb{D}, \overline{\mathbb{D}})$ (see the beginning of section 6). Also, the partial derivative of g with respect to the first variable at the origin is just $f'(0)$ and the partial derivative of g with respect to the second variable at the origin is $f(0)$ (see section 5). Therefore,

$$E_M((0,0); v) = \sup\{|v_1 f'(0) + v_2 f(0)| : f \in \mathcal{O}(\mathbb{D}, \overline{\mathbb{D}})\}$$

The set of pairs $(f'(0), f(0))$ as f varies over $\mathcal{O}(\mathbb{D}, \overline{\mathbb{D}})$ is really just the pairs (a, b) where $|a| + |b|^2 \leq 1$, by the Schwarz-Pick Lemma. This reduces the problem to maximizing $|v_1|s + |v_2|t$ over all $s, t \in [0, 1]$ satisfying $s + t^2 \leq 1$. The function we are maximizing is linear, so the maximum occurs on the boundary. Therefore, the problem is just a matter of finding the maximum of $|v_1|(1-t^2) + |v_2|t$ for $0 \leq t \leq 1$. Hence,

$$E_M((0,0); v) = \begin{cases} |v_2| & \text{if } |v_2| \geq 2|v_1| \\ \frac{4|v_1|^2+|v_2|^2}{4|v_1|} & \text{if } |v_2| < 2|v_1| \end{cases}$$

as desired.

Next, let $x = (a,b) \in M \setminus \{(0,0\}$ and define $v = (3a, 2b)$. The Carathéodory pseudo-metric at (a,b) is

$$E_M(x;v) = \sup\left\{\frac{|dg_x v|}{1-|g(x)|^2} : g \in \mathcal{O}(M,\mathbb{D})\right\}$$

If $\lambda = a/b$ and $h(\zeta) = g(\zeta^3, \zeta^2)$, then $v = \lambda(3\lambda^2, 2\lambda)$ and since $dg_x(3\lambda^2, 2\lambda) = h'(\lambda)$ we see that

$$E_M(x;v) = |\lambda| \sup\{\rho(h(\lambda); h'(\lambda)) : h \in \mathcal{O}(\mathbb{D},\mathbb{D}) \text{ and } h'(0) = 0\}$$

As in the case of the Carathéodory pseudo-distance we can assume $h(0) = 0$ and therefore h has the form $h(\zeta) = \zeta^2 f(\zeta)$ for some $f \in \mathcal{O}(\mathbb{D}, \overline{\mathbb{D}})$. Hence,

$$E_M(x;v) = |\lambda| \sup\left\{\frac{|2\lambda f(\lambda) + \lambda^2 f'(\lambda)|}{1-|\lambda|^4|f(\lambda)|^2} : f \in \mathcal{O}(\mathbb{D},\overline{\mathbb{D}})\right\}$$

Like before, $(f(\lambda), f'(\lambda))$ varies over all pairs (A,B) satisfying $|B|(1-|\lambda|^2) \leq 1-|A|^2$. This reduces the problem to maximizing

$$\frac{2|\lambda|s + |\lambda|^2 t}{1-|\lambda|^4 s^2}$$

over the set of non-negative s,t satisfying $s^2 + t(1-|\lambda|^2) \leq 1$. It is easy to check that the maximum always occurs when $s = 1$ and $t = 0$. Since $\lambda^2 = b$ we see that

$$E_M(x;v) = \frac{2|b|}{1-|b|^2}.$$

9. Proof of a Mixed Carathéodory-Pick Interpolation Problem

By precomposing all functions with ϕ_{z_3} we may assume $z_3 = 0$ in proposition 4.5. Then, all functions of interest will correspond to functions in $\mathcal{O}(M,\mathbb{D})$, and therefore it is clear that if there is a function $f \in \mathcal{O}(\mathbb{D},\mathbb{D})$ which satisfies the $f'(0) = 0$, $f(z_i) = w_1$ for $i = 1,2$, then the inequality (4.6) holds.

On the other hand, if the inequality (4.6) holds (again with $z_3 = 0$), then pick a function $h \in \mathcal{O}(M,\mathbb{D})$ with

$$\rho(h(p(z_1)), h(p(z_2))) = c_M(p(z_1), p(z_2))$$

(we know such a function exists by the formula for c_M) and then set $g := h \circ p \in \mathcal{O}(\mathbb{D},\mathbb{D})$. The function g satisfies $\rho(w_1, w_2) \leq \rho(g(z_1), g(z_2))$ and

by composing g with an appropriate function we can find a function with $f(z_1) = w_1$, $f(z_2) = w_2$, and $f'(0) = 0$.

Finally, if f satisfies the interpolation problem and equality in (4.6), then $g := \phi_{f(0)} \circ f$ satisfies equality as well. Hence, when

$$\alpha_0 := \frac{1}{2}\left(\frac{1}{\bar{z}_1} + z_1 + \frac{1}{\bar{z}_2} + z_2\right)$$

is in the disk, $g(\zeta)$ is of the form $\mu\zeta^2\phi_{\alpha_0}(\zeta)$ where μ is a unimodular constant, and when $\alpha_0 \notin \mathbb{D}$, $g(\zeta)$ is of the form $\mu\zeta^2$. But, μ and $f(0)$ are uniquely determined by the fact that $w_i = \phi_{f(0)}(g(z_i))$ for $i = 1, 2$ since $g(z_1)$ and $g(z_2)$ must be distinct. So, there exists a unique automorphism of the disk ψ such that

$$f(\zeta) = \begin{cases} \psi(\zeta^2\phi_{\alpha_0}(\zeta)) & \text{if } \alpha_0 \in \mathbb{D} \\ \psi(\zeta^2) & \text{if } \alpha_0 \notin \mathbb{D} \end{cases}$$

In the first case, f is a Blaschke product of order three and in the second a Blaschke product of order two.

10. Proof of Extension Theorem

In this section we prove Theorem 4.7.

First, we need to define a few basic notions. Let X be a set. A self-adjoint function $F : X \times X \to \mathbb{C}$ (i.e. $F(x, y) = \overline{F(y, x)}$) is *positive semi-definite* if for every N and every finite subset $\{x_1, x_2, \ldots, x_N\} \subset X$ the $N \times N$ matrix with entries $F(x_i, x_j)$ is positive semi-definite. For example, by the Pick interpolation theorem the function $F : \mathbb{D}\times\mathbb{D} \to \mathbb{C}$ given by

$$F(\lambda, \delta) = \frac{1 - h(\lambda)\overline{h(\delta)}}{1 - \lambda\bar{\delta}}$$

is positive semi-definite for any $h \in \mathcal{O}(\mathbb{D}, \overline{\mathbb{D}})$.

The Pick interpolation theorem on the bidisk (see [1] page 180) can be stated as a theorem about extensions of bounded analytic functions in the following way. Given a subset X of the bidisk, and a function $g : X \to \mathbb{D}$ there exists $G \in \mathcal{O}(\mathbb{D}^2, \mathbb{D})$ with $G|_X = g$ if and only if there exist positive semi-definite functions Δ and Γ on $X \times X$ such that for each $z = (z_1, z_2), w = (w_1, w_2) \in X$

$$1 - g(z)\overline{g(w)} = \Gamma(z, w)(1 - z_1\bar{w}_1) + \Delta(z, w)(1 - z_2\bar{w}_2)$$

We should mention that the portion of this theorem which we shall use (namely sufficiency) has a quite simple proof—it is an application of the so-called "lurking isometry" technique.

To prove theorem 4.7 suppose $f \in \mathcal{O}(M, \mathbb{D})$ and $f(0,0) = 0$. Then, as in earlier arguments, $(f \circ p)(\zeta) = f(\zeta^3, \zeta^2) = \zeta^2 h(\zeta)$ for some $h \in \mathcal{O}(\mathbb{D}, \overline{\mathbb{D}})$. For any $\delta, \lambda \in \mathbb{D}$, we have

$$\begin{aligned} 2-&f(p(\lambda))\overline{f(p(\delta))} = (1-\lambda^3\bar{\delta}^3) \\ &+ \left(1 + \lambda^2\bar{\delta}^2 \frac{1-h(\lambda)\overline{h(\delta)}}{1-\lambda\bar{\delta}} + \frac{\lambda^3\bar{\delta}^3 h(\lambda)\overline{h(\delta)}}{1-\lambda^2\bar{\delta}^2}\right)(1-\lambda^2\bar{\delta}^2) \end{aligned}$$

Therefore, for $z = (z_1, z_2), w = (w_1, w_2) \in M$

$$2 - f(z)\overline{f(w)} = \Gamma(z,w)(1 - z_1\bar{w}_1) + \Delta(z,w)(1 - z_2\bar{w}_2) \tag{10.1}$$

where $\Gamma(z,w) = 1$ and

$$\Delta(z,w) = 1 + z_1\bar{w}_1 \frac{1 - h(q(z))\overline{h(q(w))}}{1 - q(z)\overline{q(w)}} + \frac{z_2\bar{w}_2 h(q(z))\overline{h(q(w))}}{1 - z_1\bar{w}_1}$$

(recall $q(z) = z_1/z_2$ for $z \neq (0,0)$ and $q(0,0) = 0$). Now, Γ is clearly positive semi-definite, and Δ is positive semi-definite because of the fact that positive semi-definite functions are closed under addition and multiplication (by the Schur product theorem) and by the Pick interpolation theorem on the disk (applied to h). This proves f has an extension to the bidisk with supremum norm at most $\sqrt{2}$ (by dividing through (10.1) by 2).

To prove any holomorphic function $f \in \mathcal{O}(M, \mathbb{D})$ (regardless of its value at the origin) can be extended to the bidisk with supremum norm at most $2\sqrt{2} + 1$, simply apply the result just proved to $(f - f(0))/2$.

Finally, the function

$$\tilde{g}(\lambda) = \lambda^2 \frac{0.5 - \lambda}{1 - 0.5\lambda}$$

corresponds to a function $g \in \mathcal{O}(M, \mathbb{D})$ with $g(\lambda^3, \lambda^2) = \tilde{g}(\lambda)$. The partial derivatives of g at $(0,0)$ are just the coefficients of λ^3 and λ^2 in the power series for $\tilde{g}$; i.e. they are -0.75 and 0.5. Suppose G is a bounded extension of g to the bidisk with sup norm R. Then, by the Schwarz lemma on the bidisk

$$0.75/R + 0.5/R \leq 1$$

which implies $R \geq 5/4$, as desired.

References

[1] J. Agler and J.E. McCarthy, *Pick Interpolation and Hilbert Function Spaces*, A.M.S., Providence, RI, 2002.

[2] _______, *Norm Preserving Extensions of Holomorphic Functions from Subvarieties of the Bidisk*, Ann. of Math., **157** (2003), 289-312.

[3] C. Carathéodory and L. Fejér, *Über den Zusammenhang der Extremen von harmonischen Funktionen mit ihren Koeffizienten und über den Picard-Landauschen Satz*, Rend. Circ. Mat. Palermo. **32** (1911), 218-239.

[4] R. Gunning, *Introduction to Holomorphic Functions of Several Variables, Vol II*, Wadsworth Inc., California, 1990.

[5] _______, *Introduction to Holomorphic Functions of Several Variables, Vol III*, Wadsworth Inc., California, 1990.

[6] A. Isaev, S. Krantz, *Invariant Distances and Metrics in Complex Analysis*, Notices of the American Math. Society, **47** (2000), 546-553.

[7] M. Jarnicki, P. Pflug, *Invariant Distances and Metrics in Complex Analysis*, de Gruyter Expositions in Mathematics 9, Walter de Gruyter, 1993.

[8] _______, *Invariant Distances and Metrics in Complex Analysis - Revisited*, Diss. Math. **430** (2005), 1-192.

[9] S. Kobayashi, *Hyperbolic Complex Spaces*, Springer Verlag, 1998.

[10] L. Lempert, *La métrique de Kobayashi et la représentation des domaines sur la boule*, Bull. Soc. Math. France **109** (1981), 427-474.

[11] R. Nevanlinna, *Über beschränkte Funktionen*, Ann. Acad. Sci. Fenn. Ser. A **13** (1919), no. 1.

[12] G. Pick, *Über die Beschränkungen analytischer Funktionen, welche durch vorgegebene Funktionswerte bewirkt werden*, Math. Ann. **77** (1916), 7-23.

[13] P.L. Polyakov and G.M. Khenkin, *Integral Formulas for the $\bar{\partial}$-Equation and an Interpolation Problem in Analytic Polyhedra*, Trans. Moscow Math. Soc. **53** (1991), 135-175.

[14] J. Wallis, *Arithmetica Infinitorum*, 1655.

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